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EFFECT OF HORIZONTAL AND VERTICAL SIDE FORCES AND MOMENTS ON STABILITY OF A SYMMETRIC MISSILE IN ASCENDING OR DESCENDING FLIGHT

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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Lloyd and Brown have shown that constant horizontal and vertical side forces and moments applied to a spinning projectile can result in dynamic instability. This instability arises from the nonlinear terms in the fixed-plane coordinate system spin that appear in the equations of motion. By quasilinearization, these nonlinear terms are shown to affect the frequencies slightly but the damping rates not at all. By the use of the fixed-plane system (rather than Lloyd and Brown's nonrolling system) simpler relations can be developed, the full effects of gravity, drag and roll damping obtained and the limitation to (over)

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Block 20 (Abstract)(continued):

large gyroscopic stability factors removed. Very simple stability bounds are given for a slowly spinning or nonspinning finned missile such as Copperhead.

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I. INTRODUCTION

In 1977, Lloyd and Brown[1] investigated the feasibility of controlling a 105mm spinning projectile by means of horizontal and vertical side forces. Their numerical calculations yielded the surprising result that an applied constant-amplitude yaw moment could cause dynamic instability. The usual linear analysis seems to predict that such a moment would cause a steady-state horizontal trim angle but would have no effect on the dynamic stability.

This difficulty was resolved by Lloyd and Brown through the observation that the differential equation for the angular motion in fixed-plane coordinates* contained nonlinear terms in $\dot{\phi}_{FP}$, the coordinate system spin rate. These usually neglected terms vanish completely when the equations are transformed to nonrolling coordinates. The terms involving the horizontal and vertical control moments become nonlinear terms that can be easily linearized. The resulting sixth-order system can be approximately solved for large gyroscopic stability factor ($s_g > 4$) and excellent agreement with the numerical results obtained. The theory, however, only partially considers the influence of gravity and neglects the effect of drag and roll damping moment.

In this report, we will show that the coordinate system transformation is unnecessary and that a proper linearization in the fixed-plane coordinates requires the solution of a much simpler fourth-order system. This allows the very easy inclusion in the theory of the full effect of gravity as well as the effects of drag and roll damping. Much more importantly, the requirement of high stability factor is eliminated so that the very important case of a finned missile with little or no spin can be studied. Finally, the effective technique of quasi-linearization will be used to derive the effect of the truly nonlinear part of the $\dot{\phi}_{FP}$ terms on the frequencies and the damping rates of the motion.

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1. *K.H. Lloyd and D.P. Brown, "Influence of Gravity and Applied Side Forces on the Stability of a Spinning Projectile," Weapons Research Establishment TR 1906(W), South Australia, November 1977, AD A053648. (See also "Instability of Spinning Projectiles During Terminal Guidance,"* *Journal of Guidance and Control* *2, Jan-Feb 1979, pp. 65-70.)*

**Fixed plane axes $\hat{x}$, $\hat{y}$, $\hat{z}$ pitch and yaw with the missile but roll so that the $\hat{y}$-axis is always in the horizontal plane.*

II. EQUATIONS OF MOTION

The plane trajectory of a particle flying at velocity V and trajectory angle θ_T with respect to the horizontal can be described by the equations[2]

$$\frac{V'}{V} = - C_D^* - g^* \tag{2.1}$$

$$\theta_T' = - g \ell V^{-2} \cos \theta_T \tag{2.2}$$

where

$$C_D^* = \frac{\rho S \ell}{2m} C_D$$

$$g^* = g \ell V^{-2} \sin \theta_T$$

and where derivatives are with respect to the nondimensional arc-length, s. These equations are good approximations for the actual variation of V and θ_T for a symmetric missile. The moments and transverse forces that have a measurable effect on the missile's motion are usually expressed in missile-fixed coordinates[3] as

$$M_X = \left(\frac{1}{2}\right) \rho S \ell V^2 \left[\delta_f C_{\ell_\delta} + \phi' C_{\ell_p} \right] \tag{2.3}$$

$$\begin{aligned} M_Y + i M_Z = \left(\frac{1}{2}\right) \rho S \ell V^2 \Bigg[& \left(\phi' C_{M_{p\alpha}} - i C_{M_\alpha} \right) \xi \\ & + C_{M_q} \mu - i C_{M_{\dot\alpha}} \left(\xi' + i \phi' \xi \right) \Bigg], \end{aligned} \tag{2.4}$$

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2. C.H. Murphy, *"Gravity-Induced Angular Motion of a Spinning Missile," Ballistic Research Laboratories Report No. 1546, July 1971, AD 730641. (See also* ***Journal of Spacecraft and Rockets 8****, August 1971, pp. 824-828.)*

3. *C.H. Murphy, "Free Flight Motion of Symmetric Missiles," Ballistic Research Laboratories Report No. 1216, July 1963, AD 442757.*

$$F_Y + i\,F_Z = -\left(\frac{1}{2}\right) \rho\, S\, V^2\, C_{N_\alpha}\, \xi \qquad (2.5)$$

where

$$\phi' = p\, \ell\, V^{-1}$$

$$\xi = (v + i\,w)\, V^{-1}$$

$$\mu = (q + i\,r)\, \ell\, V^{-1}$$

The complex variable ξ locates the plane of the velocity vector and has a magnitude that is the sine of the total angle of attack.

The roll equation can be obtained for the roll moment of Equation (2.3) and differs from the usual roll equation[3] by a gravity term that acts on the dynamic pressure:

$$\phi'' = (D_1 + g^*)\, \phi' + D_2 \qquad (2.6)$$

where

$$D_1 = C_D^* + k_a^{-2}\, C_{\ell_p}^*$$

$$D_2 = k_a^{-2}\, \delta_f\, C_{\ell_\delta}^*$$

$$k_a = [I_x/m\, \ell^2]^{1/2}$$

In addition to the aerodynamic force and moment and the gravity force, we assume constant amplitude control forces and moments that are perpendicular to the projectile's axis and either in the horizontal plane or the vertical plane containing the missile's axis. These control forces, which could be produced by roll-stabilized canards, make fixed-plane coordinates most suitable for the analysis. Since fixed-plane axes pitch and yaw with the projectile but roll so that the $\hat{y}$ axis is always horizontal, it can be shown that this system has an angular velocity vector $\hat{\Omega}_{FP}$ with components[4]:

$$\vec{\Omega}_{FP} = (\dot{\phi}_{FP},\, \hat{q},\, \hat{r}) \qquad (2.7)$$

4. *C.H. Murphy, "Measurement of Nonlinear Forces and Moments by Means of Free Flight Tests," Ballistic Research Laboratories Report No. 974, February 1956, AD 93521.*

where $\dot{\phi}_{FP} = - \hat{r} \tan \theta$

θ is the angle between the missile's axis and the horizontal ($\dot{\theta} = \hat{q}$);

$\hat{q}$, $\hat{r}$ are the pitch and yaw rates in the fixed-plane coordinates

For this coordinate system, the derivatives of the linear and angular momentum can be computed in the usual way[3,4] and set equal to the sum of the external forces and moments. The equations for the transverse components can then be given in the form of two first-order complex differential equations:

$$\hat{\xi}' - i \gamma \hat{\mu} = - \gamma C^*_{L_\alpha} \hat{\xi} + (i \cos \theta + \hat{\xi} \sin \theta_T) g \ell V^{-2} + (F_{YC} + i F_{ZC}) \ell (m V^2)^{-1} - i \phi'_{FP} \hat{\xi} \tag{2.8}$$

$$\begin{aligned} \hat{\mu}' - i P \hat{\mu} = {} & k_t^{-2} \left[\phi' C^*_{M_{p\alpha}} - i C^*_{M_\alpha} \right] \hat{\xi} \\ & + \left[k_t^{-2} C^*_{M_q} + C^*_D + g^* \right] \hat{\mu} \\ & - i k_t^{-2} C^*_{M_{\dot{\alpha}}} (\hat{\xi}' + i \phi'_{FP} \hat{\xi}) \\ & + k_t^{-2} (M_{YC} + i M_{ZC})(m V^2)^{-1} - i \phi'_{FP} \hat{\mu} \end{aligned} \tag{2.9}$$

where

$P = I_x \phi' / I_y$

$\gamma = u V^{-1}$ = the cosine of the total angle of attack

The starred coefficients are of the order 10^{-4} while the dimensionless control forces and moments will be limited to at most 10^{-4}. Thus, products of these terms can be neglected when $\hat{\mu}$ is eliminated between Equations (2.8) and (2.9) to obtain:

$$\hat{\xi}'' + [H - g^* - \frac{\gamma'}{\gamma} - i\ P]\hat{\xi}' - [M + i\ P\ T]\hat{\xi} = \Phi + \hat{G} + C \qquad (2.10)$$

where

$$\Phi = -\,2\ i\ \phi'_{FP}\ \hat{\xi}' - \left\{ \phi'_{FP}\ (P - \phi'_{FP}) + i\ [\phi'_{FP}\ (H - \frac{\gamma'}{\gamma}) + \phi''_{FP}] \right\} \hat{\xi}$$

$$\hat{G} = [P \cos\theta - i\ \hat{\mu} \sin\theta]\ g\ \ell\ V^{-2} + g^*\ (\hat{\xi}' - i\ P\ \hat{\xi})$$

$$C = i \left[\gamma\ k_t^{-2}\ (M_{YC} + i\ M_{ZC}) - (P - i\ \gamma' \gamma^{-1})\ell(F_{YC} + i\ F_{ZC}) \right] (mV^2)^{-1}$$

$$\doteq i\ \gamma\ k_t^{-2}\ (M_{YC} + i\ M_{ZC})(m\ V^2)^{-1}$$

III. LINEARIZED SOLUTION FOR CONTROL MOMENT

For simplicity, we will first neglect the gravity terms in Equation (2.10) and consider the linear approximation to Φ. In doing this, it is most important to remember that Equation (2.10) predicts a steady-state equilibrium angle

$$\hat{\xi}_e = \hat{\beta}_e + i\,\hat{\alpha}_e = \frac{-C}{M + i\,PT} \qquad (3.1)$$

If the small force terms are neglected, the real part of Equation (2.8) yields

$$\gamma\,\hat{r}\,\ell\,V^{-1} = -\hat{\beta}' + \phi'_{FP}\,\hat{\alpha} \qquad (3.2)$$

Equation (3.2) is now multiplied by tan θ and solved for ϕ'_{FP} .

$$\phi'_{FP} = \frac{\hat{\beta}'\,\tan\theta}{\gamma + \hat{\alpha}\,\tan\theta} \qquad (3.3)$$

Now

$$\theta = \int q\,dt \doteq \theta_e + (\hat{\alpha} - \hat{\alpha}_e)\gamma^{-1} \qquad (3.4)$$

$$\therefore \phi'_{FP} = a\,\hat{\beta}' \Big\{ \tan\theta_e + \gamma_e^{-1}(\hat{\alpha} - \hat{\alpha}_e) + \gamma_e^{-1}(a\,\hat{\beta}_e \tan\theta_e)(\hat{\beta} - \hat{\beta}_e) + a_1(\hat{\alpha} - \hat{\alpha}_e)^2 + a_2(\hat{\alpha} - \hat{\alpha}_e)(\hat{\beta} - \hat{\beta}_e) + a_3(\hat{\beta} - \hat{\beta}_e)^2 \Big\}$$

$$\doteq a\,\beta'\,\tan\theta_e \qquad (3.5)$$

where

$$a = [\gamma_e + \hat{\alpha}_e \tan\theta_e]^{-1}$$

$$a_1 = [2\hat{\alpha}_e - (1 - \hat{\beta}_e^2)\,a\tan\theta_e](2\gamma_e^3)^{-1}$$

$$a_2 = (1 + a\,\gamma_e)\hat{\beta}_e\,\gamma_e^{-3}$$

$$a_3 = (1 - \hat{\alpha}_e^2 + 2a\,\gamma_e\,\hat{\beta}_e^2)(a\tan\theta_e)(2\,\gamma_e^3)^{-1}$$

Since a linear analysis is concerned with small amplitude motion about the equilibrium angle $\hat{\xi}_e$, Φ is expanded in powers of $\hat{\xi} - \hat{\xi}_e$ and its derivatives. The linear part of this expansion is:

$$\Phi_\ell = -\,i\,a\tan\theta_e\,[\hat{\beta}'' + (H - i\,P)\hat{\beta}']\hat{\xi}_e \qquad (3.6)$$

The usual solution to the linearized Equation (2.10) neglecting Φ is

$$\hat{\xi} = \hat{\xi}_e + K_1 e^{i\phi_1} + K_2 e^{i\phi_2} \tag{3.7}$$

where

$$\phi_j' = (1/2)(P \pm \sqrt{P^2 - 4M})$$

$$K_j'/K_j = \lambda_j = \frac{-\phi_j' H + PT - \phi_j''}{2 \phi_j' - P}$$

It is important to note that $|\phi_j'| \sim 10^{-2}$ and $|\lambda_j| \sim 10^{-4}$. For ascending or descending flight, Φ_ℓ introduces terms in

$$\hat{\beta}' = (\hat{\xi}' + \bar{\hat{\xi}}')/2 \text{ and } \hat{\beta}'' = (\hat{\xi}'' + \bar{\hat{\xi}}'')/2 .$$

The linearized Equation (2.10) becomes

$$\left[1 + (a\ i\ \hat{\xi}_e/2) \tan \theta_e\right]\left[\hat{\xi}'' + (H - i\ P)\ \hat{\xi}'\right] - (M + i\ P\ T)\hat{\xi}$$
$$= C - (a\ i\ \bar{\hat{\xi}}_e/2) \tan \theta_e \left[\bar{\hat{\xi}}'' + (H - i\ P)\ \bar{\hat{\xi}}'\right] \tag{3.8}$$

As is shown in Reference 5, the effect of the conjugate terms in Equation (3.8) is to add two additional modes in $-\phi_1'$ and $-\phi_2'$. For reasonable values of $\hat{\xi}_e$, the amplitude of these modes will be much less than K_1 and K_2. If we approximate the actual solution by the two-mode solution of Equation (3.7) and substitute in Equation (3.8), the conjugate terms have no contribution to the damping or

5. *C.H. Murphy, "Angular Motion of Spinning Almost Symmetric Missiles," U.S. Army Armament Research and Development Command, Ballistic Research Laboratory Technical Report 02121, November 1978, AD A063538.*

frequency equations obtained from the coefficients of $\exp(i\,\phi_j)$. For simplicity, we make the usual size assumptions $\left(|\lambda_j| \ll |\phi_j'|,\ |H| \ll |\phi_j'|,\ |T| \ll |\phi_j'|\right)$ and retain only terms linear in $\hat{\alpha}_e$ and $\hat{\beta}_e$.

$$(\phi_j')^2 - \phi_j' P + M + \tan\theta_e\,[M\,\hat{\alpha}_e + P\,T\,\hat{\beta}_e]/2 = 0 \qquad (3.9)$$

$$\lambda_j = \frac{-H\,\phi_j' + P\,T - \phi_j'' - \tan\theta_e\,[M\,\hat{\beta}_e - P\,T\,\hat{\alpha}_e]/2}{2\,\phi_j' - P} \qquad (3.10)$$

IV. LINEARIZED SOLUTION FOR GRAVITY

For no control forces (C = 0), Equation (2.10) predicts a steady-state equilibrium angle

$$\hat{\xi}_e = \frac{-G}{M + i\,P(T - g^*)} \doteq -G/M \qquad (4.1)$$

where

$$G = P\,g\,\ell\,V^{-2}\cos\theta_e\ .$$

Since $\hat{\alpha}_e$ is zero, $\theta_T = \theta_e$ and the linearized $\hat{G}$ and Φ become

$$\hat{G} = G - i\,g^*\,P\,\hat{\beta} \qquad (4.2)$$

$$\Phi = i\,g^*\,(P/M)\,[\hat{\beta}'' + (H - i\,P)\hat{\beta}'] \qquad (4.3)$$

Equation (2.10) reduces to

$$\begin{aligned}\hat{\xi}'' + (H - g^* - i\,P)\hat{\xi}' - (M + i\,P\,T)\hat{\xi}& \\ = G + i\,g^*\,(P/M)\,[\hat{\beta}'' + (H - i\,P)\hat{\beta}' - M\,\hat{\beta}]&\end{aligned} \qquad (4.4)$$

The solution to Equation (4.4) can be approximated by the two-mode Equation (3.7) with the result that the second term on the right of Equation (4.4) has no measurable contribution to the frequency or damping.

$$(\phi_j')^2 - P\,\phi_j' + M = 0 \tag{4.5}$$

$$\lambda_j = \frac{-(H - g^*)\,\phi_j' + P\,T - \phi_j''}{2\,\phi_j' - P} \tag{4.6}$$

Differentiating Equation (4.5),

$$\phi_j'' = \frac{P'\,\phi_j' - M'}{2\,\phi_j' - P} \tag{4.7}$$

Since D_2 in Equation (2.6) is zero for a body of revolution, the damping rate for a shell becomes

$$\lambda_j = \frac{-H\,\phi_j + P\,T}{2\,\phi_j' - P} - \frac{[D_1\,P + 2\,\phi_j'\,(P - \phi_j')\,g^* - M']}{(2\,\phi_j' - P)^2} \tag{4.8}$$

It is interesting to note that as the gyroscopic stability increases, ϕ_1' goes to P and ϕ_2' goes to 0 and the contribution of gravity to damping decays to zero.

Equation (4.6) is precisely the same as that in Reference 2. The derivation given in Reference 2, however, separately neglected the second term in Equation (4.2) and all of Φ . The correct derivation combines these terms and shows that their combined effect can be neglected.

V. QUASILINEAR ANALYSIS OF Φ

For horizontal flight without control forces or gravity, Φ is cubic and was shown in Reference 4 to cause a change in frequency. Numerical calculations by Clark and Hodapp[6] showed that this frequency shift was very well predicted by the quasilinear analysis. In this section, we will derive the quasilinear prediction for cubic Φ and ascending or descending flight with control forces but no gravity.

The cubic part of the nonlinear term in γ' on the left side of Equation (2.10) can be easily computed:

$$(\gamma'/\gamma)\ \hat{\xi}' = \frac{1}{2}\ \frac{(\gamma^2)'}{\gamma^2}\ \hat{\xi}'$$

$$\doteq -\frac{1}{2}\ (\hat{\xi}\ \bar{\hat{\xi}})'\ \hat{\xi}' \tag{5.1}$$

The quasilinear technique then assumes an undamped motion of the form of Equation (3.7) and seeks the average in-phase and out-of-phase contributions to the coefficient of $K_j \exp(i\ \phi_j)$ by the relation

$$[F]_{j,av} = \frac{1}{K_j\ S_W} \int_0^{S_W} F\ e^{-i\ \phi_j}\ ds \tag{5.2}$$

If there is no equilibrium angle, S_W is $2\pi\ (\phi_1' - \phi_2')^{-1}$, the wavelength of $\hat{\xi} \exp(-i\phi_j)$. For nonzero equilibrium angles, the integrand has several wavelengths present and S_W is taken to be large compared to the largest of these.

Now

$$[\hat{\xi}\ \bar{\hat{\xi}}]' = i\Big[\hat{\phi}'\ K_1 K_2 (e^{i\hat{\phi}} - e^{-i\hat{\phi}})$$

$$+ \phi_1'\ K_1\ \delta_c\left(e^{i\phi_1} - e^{-i\phi_1}\right) + \phi_2'\ K_2\ \delta_c\left(e^{i\phi_2} - i\ \phi_2\right)\Big] \tag{5.3}$$

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6. *E.L. Clark, Jr., and A.E. Hodapp, "An Improved Technique for Determining Missile Roll Rate with the Epicyclic Theory," Sandia Laboratories SC-DC-70-4768, April 1970.*

where

$$\delta_c = |\hat{\xi}_e|$$

$$\hat{\phi} = \phi_1 - \phi_2 \quad .$$

Therefore,

$$[\gamma' \ \gamma^{-1} \ \hat{\xi}']_{1,av} = (1/2) \left\{ \phi'_2 \ (\phi'_1 - \phi'_2) \ K_2^2 + (\phi'_2)^2 \ K_2^2 \ \delta_c \ K_1^{-1} \left[e^{i(2 \phi_2 - \phi_1)} \right]_{av} \right\} \tag{5.4}$$

where

$$[\ ]_{av} = \frac{1}{S_W} \int_0^{S_W} [\ ] \, ds$$

For a special value of spin, $2 \phi'_2 = \phi'_1$ and the last term in Equation (5.4) has a nonzero contribution. If we assume that this special value of spin does not occur,

$$[\gamma' \gamma^{-1} \ \hat{\xi}']_{1,\ av} = \phi'_2 \ (\phi'_1 - \phi'_2) \ K_2^2/2 \tag{5.5}$$

$$[\gamma' \gamma^{-1} \ \hat{\xi}']_{2,\ av} = \phi'_1 \ (\phi'_2 - \phi'_1) \ K_1^2/2 \tag{5.6}$$

A similar calculation can be done for the nonlinear part of Φ . The combined nonlinear contributions to the first mode have the form*:

$$[\gamma' \gamma^{-1} \hat{\xi}' + \Phi - \Phi_\ell]_{1,av} = b_1 + (\text{terms that are zero when } \hat{\alpha}_e = \hat{\beta}_e = 0) \tag{5.7}$$

where

$$b_1 = \left[\gamma' \gamma^{-1} \hat{\xi}'\right]_{1,av} + a^2 \tan^2 \theta_e \left[(\hat{\beta}')^2 (\hat{\xi} - \hat{\xi}_e)\right]_{1,av}$$

$$- \frac{2 i a}{\gamma_e} \left[\beta' \xi' (\hat{\alpha} - \hat{\alpha}_e)\right]_{1,av}$$

$$- \frac{a P}{\gamma_e}\left[\beta' (\hat{\alpha} - \hat{\alpha}_e)(\xi - \xi_e)\right]_{1,av} - \frac{i a}{\gamma_e}\left[\beta'' (\hat{\alpha} - \hat{\alpha}_e)(\xi - \xi_e)\right]_{1,av}$$

$$= (1/2)\left[(\phi_1')^2 K_1^2 + P \phi_2' K_2^2\right] a^2 \tan^2 \theta_e$$

$$- (2 \gamma_e)^{-1} (\phi_1' - \phi_2') \left[a \phi_1' K_1^2 + (a - \gamma_e) \phi_2' K_2^2\right]$$

A similar expression for the second mode can be obtained by interchanging subscripts 1 and 2.

These average nonlinear contributions can now be added to the coefficients of $\exp(i\phi_j)$ in the usual derivation of the linear damping rates and frequencies. For the special case of no control forces and moments, the damping rates are unaffected and the frequency equations become

$$(\phi_1')^2 - P \phi_1' + [M]_{1,av} + \left\{\tan^2 \theta_e \left[(\phi_1')^2 K_1^2 + P \phi_2' K_2^2\right] + (\phi_2' - \phi_1') \phi_1' K_1^2\right\} \Big/ 2 = 0 \tag{5.8}$$

For simplicity, the quite small nonlinear terms in H have been omitted.

Table I. Assumed Parameters for 105mm Shell

ℓ	=	.105 m	C_D	=	0.13
S	=	.0087 m^2	C_{L_α}	=	1.7
m	=	15 kg	C_{ℓ_p}	=	- 0.012
I_x	=	.023 kg-m^2	C_{M_α}	=	3.8
I_y	=	.22 kg-m^2	$C_{M_{p\alpha}}$	=	0.2
ρ	=	1.05 kg/m^3	$C_{M_q} + C_{M_{\dot\alpha}}$	=	- 8
V	=	250 m/s			
p	=	1050 rad/s			

and a similar equation for the other mode. These frequency equations, for $\theta_e = 0$ and an appropriate choice for M, were the equations that gave the excellent agreement with the numerical calculations of Reference 6.

VI. DISCUSSION

According to our analysis, the effect of gravity on damping rates is to replace H by H-g* . Equation (3.10) for the damping rates becomes

$$\lambda_j = \frac{-(H - g^*)\,\phi_j' + P\,T - \phi_j'' - (\tan\theta_e)\left[M\,\hat{\beta}_e - P\,T\,\hat{\alpha}_e\right]/2}{2\,\phi_j' - P} \tag{6.1}$$

Since $|P\ T|$ is usually much smaller than $|M|$, the effect of $\hat{\alpha}_e$ on the damping rates can be neglected. Equation (6.1) can therefore be used to derive stability boundaries for the maximum trim angles for $\hat{\beta}$. For a gyroscopically stable missile with positive spin ($\phi_1' > P/2 > \phi_2' > 0$),

$$B_1 < \hat{\beta}_e \tan\theta_e < B_2 \tag{6.2}$$

where

$$B_j = -(2/M)\left[(H - g^*)\,\phi_j' - P\,T + \phi_j''\right]$$

Table I gives the various parameters for a 105mm shell and Figure 1 gives the boundaries B_1 and B_2 in degrees as functions of the gyroscopic stability factor, s_g . The Lloyd and Brown results were limited to large stability factors and their numerical calculations were for $s_g = 5.8$. As we can see from Figure 1, the allowable range for $\hat{\beta}_e$ becomes quite small for stability factors less than 2 .

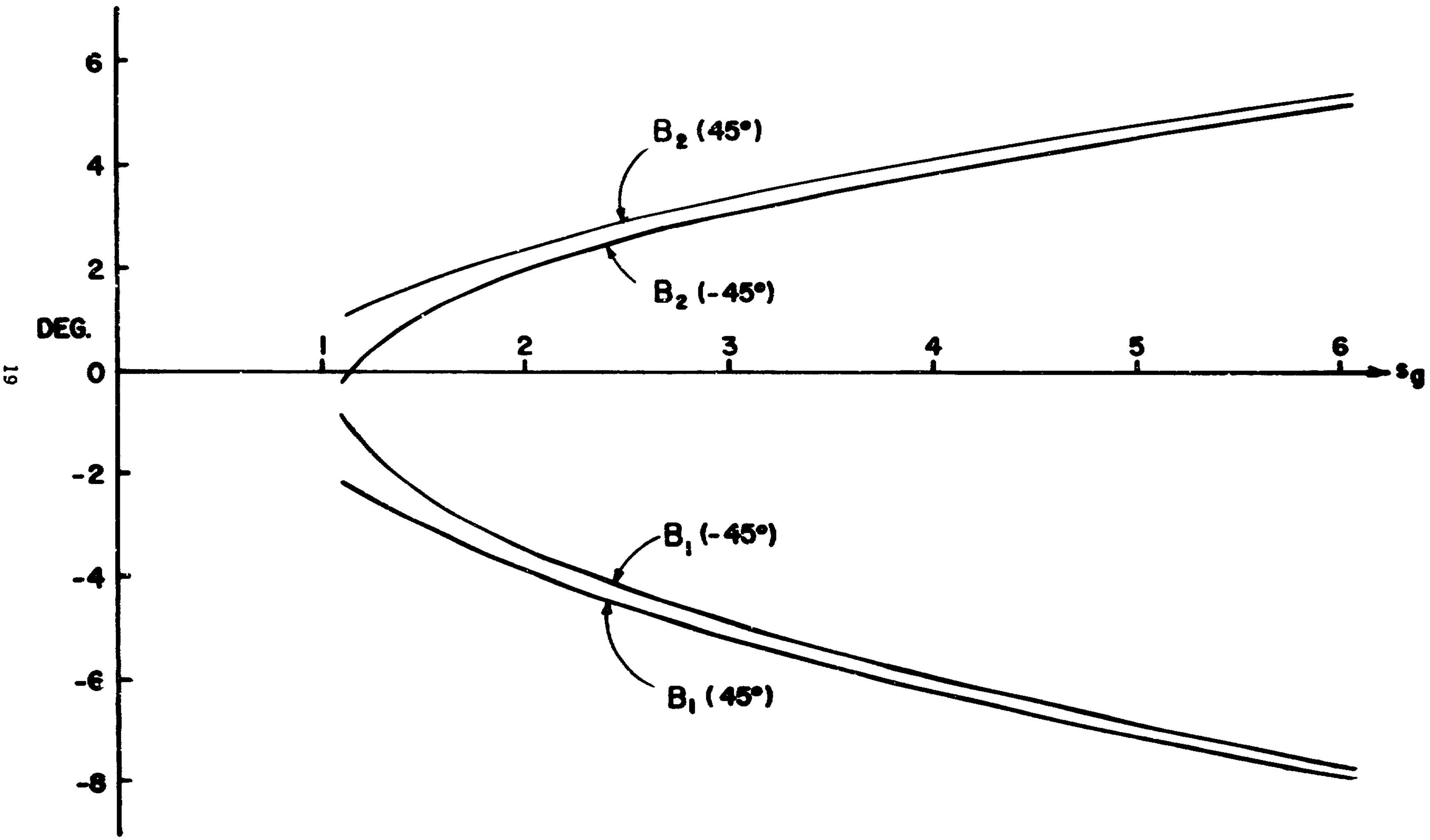

Figure 1. B_j versus s_g for $\theta_T = 45^0, -45^0$

A particularly interesting case is that of a statically stable missile with little or no spin ($M < 0$, $|s_g| << 1$) . For this case,

$$\frac{\phi_j''}{M} = \frac{-M'/M}{2\,\phi_j'} \tag{6.3}$$

For an exponential air density and constant static moment coefficient,

$$M'/M = \rho'/\rho = -\,\sigma\,\ell\,\sin\theta_T \tag{6.4}$$

where

$$\rho = \rho_0\, e^{-\sigma z}$$

Since $\phi_j' \doteq \pm\sqrt{-M}$, the stability bounds reduce to the very special condition:

$$|\hat{\beta}_e \tan\theta_e| < B \tag{6.5}$$

where

$$B = \frac{2(H - g^*) - \sigma\,\ell\,\sin\theta_T}{\sqrt{-M}} \quad .$$

It is interesting to note that the value of B for the Copperhead missile is estimated to be 12 degrees.

REFERENCES

1. K.H. Lloyd and D.P. Brown, "Influence of Gravity and Applied Side Forces on the Stability of a Spinning Projectile," Weapons Research Establishment TR 1906(W), South Australia, November 1977, AD A053648. (See also "Instability of Spinning Projectiles During Terminal Guidance," *Journal of Guidance and Control* 2, Jan-Feb 1979, pp. 65-70.)

2. C. H. Murphy, "Gravity-Induced Angular Motion of a Spinning Missile," Ballistic Research Laboratories Report No. 1546, July 1971, AD 730641. (See also *Journal of Spacecraft and Rockets* 8, August 1971, pp. 824-828.)

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LIST OF SYMBOLS

a	$[\gamma_e + \hat{\alpha}_e \tan \theta_e]^{-1}$
B	$\lvert B_j \rvert$ for a statically stable, nonspinning missile
B_1, B_2	lower and upper bounds on $\hat{\beta}_e \tan \theta_e$
C	that part of the fixed-plane complex yaw forcing function due to the control force and moment
C_D	$\dfrac{\text{drag force}}{(1/2)\ \rho\ S\ V^2}$
C_{L_α}	$\dfrac{\text{lift force}}{(1/2)\ \rho\ S\ V^2\ \lvert\xi\rvert}$
C_{ℓ_p}	roll damping moment coefficient
C_{ℓ_δ}	roll moment coefficient due to canted fins
$C_{M_{p\alpha}}$	$\dfrac{\text{Magnus moment}}{(1/2)\ \rho\ S\ \ell\ V^2\ \phi'\ \lvert\xi\rvert}$
$C_{M_q} + C_{M_{\dot{\alpha}}}$	$\dfrac{\text{sum of the damping moments}}{(1/2)\ \rho\ S\ \ell\ V^2\ \lvert\mu\rvert}$
C_{M_α}	$\dfrac{\text{static moment}}{(1/2)\ \rho\ S\ \ell\ V^2\ \lvert\xi\rvert}$
C_{N_α}	$\gamma\ C_{L_\alpha} + C_D$, the normal force coefficient
D_1	$C_D^* + k_a^{-2}\ C_{\ell_p}^*$
D_2	$k_a^{-2}\ \delta_f\ C_{\ell_\delta}^*$

LIST OF SYMBOLS
(Continued)

F_Y, F_Z	transverse missile-fixed components of the aerodynamic force
F_{YC}, F_{ZC}	transverse fixed-plane components of the control force
G	$P\, g\, \ell\, V^{-2} \cos \theta_e$
$\hat{G}$	that part of the fixed-plane complex yaw forcing function due to gravity
g	magnitude of the gravity acceleration
g^*	$g\, \ell\, V^{-2} \sin \theta_T$
H	$\gamma\, C^*_{L_\alpha} - C^*_D - k_t^{-2} \left(C^*_{M_q} + \gamma\, C^*_{M_{\dot\alpha}} \right)$
I_x, I_y	axial and transverse moments of inertia
K_j	magnitude of the j-th modal arm, j = 1, 2
k_a	$(I_x/m\, \ell^2)^{1/2}$
k_t	$(I_y/m\, \ell^2)^{1/2}$
ℓ	reference length
M	$\gamma\, k_t^{-2}\, C^*_{M_\alpha}$
M_X, M_Y, M_Z	missile-fixed components of the aerodynamic moment
M_{YC}, M_{ZC}	transverse fixed-plane components of the control moment
m	mass
P	$(I_x/I_y)\, \phi'$
p, q, r	missile spin, pitch and yaw rates measured in the missile-fixed system

$\hat{q}$, $\hat{r}$	missile pitch and yaw rates measured in the fixed-plane system
S	reference area
S_W	integration interval (calibers) in an averaging process
s	nondimensional arclength along the trajectory
s_g	gyroscopic stability factor
T	$\gamma\, C^*_{L_\alpha} + \gamma\, k_a^{-2}\, C^*_{M_{p\alpha}}$
t	time
u, v, w	missile-fixed components of the velocity
V	magnitude of the velocity
$\hat{x}$, $\hat{y}$, $\hat{z}$	fixed-plane axes, the $\hat{x}$-axis along the missile's longitudinal axis and the $\hat{y}$-axis always in the horizontal plane
$\hat{\alpha}$, $\hat{\beta}$	angles of attack and sideslip in the fixed-plane system
γ	$u\, V^{-1}$
δ_c	$\lvert \hat{\xi}_e \rvert$
δ_f	fin cant angle
θ	angle between the missile's axis and the horizontal
θ_T	trajectory angle
λ_j	K'_j / K_j , $j = 1, 2$
μ	$(q + i\, r)\, \ell\, V^{-1}$
ξ	$(v + i\, w)\, V^{-1}$
ρ	air density

LIST OF SYMBOLS (Continued)

ρ_0	air density at sea level
σ	1/(6700 m)
Φ	that part of the fixed-plane complex yaw forcing function due to the spin of the fixed-plane system
Φ_ℓ	linear part of Φ
ϕ'	$p \ell V^{-1}$
$\dot{\phi}_{FP}$	spin rate of the fixed-plane system
ϕ_j	orientation angle of the j-th modal arm, j = 1, 2
$\hat{\phi}$	$\phi_1 - \phi_2$
$\vec{\Omega}_{FP}$	angular velocity of the fixed-plane system

<u>Superscripts</u>

$\dot{(\ )}$	d( )/dt
$(\ )'$	$d(\ )/ds = \dot{(\ )}\ \ell\ V^{-1}$
$(\ )^*$	$\frac{\rho S \ell}{2 m}(\ )$ ... except for g*
$\hat{(\ )}$	fixed-plane value of ( )
$\bar{(\ )}$	complex conjugate of ( )

<u>Subscripts</u>

$(\ )_e$	steady-state equilibrium value
$[\ ]_{av}$	$S_W^{-1} \int_0^{S_W} [\ ]\, ds$
$[\ ]_{j,av}$	$(K_j S_W)^{-1} \int_0^{S_W} [\ ]\, e^{-i \phi_j}\, ds$

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